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Non-commutative 't Hooft instantons

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Abstract: We employ the twistor approach to the construction of U(2) multi-instantons à la 't Hooft on non-commutative $\mathbb{R}^4$. The non-commutative deformation of the Corrigan-Fairlie-'t Hooft-Wilczek ansatz is derived. However, naively substituting into it the 't Hooft-type solution is unsatisfactory because the resulting gauge field fails to be self-dual on a finite-dimensional subspace of the Fock space. We repair this deficiency by a suitable Murray-von Neumann transformation after a specific projection of the gauge potential. The proper non-commutative 't Hooft multi-instanton field strength is given explicitly, in a singular as well as in a regular gauge.

Keywords: Non-Commutative Geometry, Solitons Monopoles and Instantons, Integrable Equations in Physics.

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1. Introduction and results

It is plausible that at very small scales space-time coordinates are to be replaced by some non-commutative structure. In order to realize this idea it is necessary to merge the framework of gauge field theory with the concepts of non-commutative geometry [1, 2, 3].

The dynamics of non-abelian gauge fields involves field configurations not accessible by perturbation theory of which instantons are the most prominent (in euclidean space-time). In order to describe the non-perturbative structure of non-commutative gauge theory, it is therefore mandatory to construct the non-commutative deformation of instanton configurations.

The first examples of non-commutative instantons were given by Nekrasov and Schwarz [4] who modified the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [5] to resolve the singularities of instanton moduli space (due to zero-size instantons). Furthermore, they showed that on non-commutative $\mathbb{R}^4$ non-singular instantons exist even for the U(1) gauge group. This exemplifies the observation that noncommutativity of the coordinates eliminates singular behavior of field configurations. Since then, numerous papers have been devoted to this subject [6]–[23], mostly employing the modified ADHM construction on non-commutative euclidean space-time. Other related works have appeared in [24]–[38].

In the present paper we focus on the non-commutative generalization of 't Hooft's multi-instanton configurations for the U(2) gauge group. Nekrasov and Schwarz [4] proposed to keep the form of the 't Hooft solutions of the (commutative) self-duality equations but simply impose noncommutativity on the coordinates. This naive non-commutative 't Hooft configuration suffers from a problem, however. As was discovered by Correa et al. [18] for the spherically-symmetric one-instanton configuration, the Yang-Mills field fails to be self-dual everywhere. Technically, the deficiency originates from the appearance of a source term in the equation for the scalar field ϕ in the Corrigan-Fairlie-'t Hooft-Wilczek (CFtHW) ansatz. Here, we generalize the result of [18] to the naive non-commutative *multi*-instanton configurations by deriving their source terms and discuss the singular nature of the ansatz. For comparison, we outline the ADHM derivation of the field strength in appendix A.

In the commutative case, in contrast, such source terms are absent because the singularities of $\Box\phi$ are cancelled by the zeros of ϕ^{-1}. Yet, singularities are present in the gauge potential. However, it is well known how to remove such singularities by a singular gauge transformation, producing for example the Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton. Therefore, one may wonder if a non-commutative analogue exists which removes the source terms, thus yielding a completely *regular* non-commutative multi-instanton whose field strength is self-dual everywhere.

In the present paper, we answer this question in the affirmative. We easily adapt the *twistor approach* (which in fact underlies the ADHM scheme [5]) to the non-commutative situation, by promoting functions to operators acting on a harmonic-oscillator Fock space. Employing the simplest Atiyah-Ward ansatz for the matrix-valued function of the associated Riemann-Hilbert problem, we straightforwardly *derive* the non-commutative generalization of the CFtHW ansatz.

The shortcoming described above is remedied by projecting the naive non-commutative 't Hooft multi-instanton field strength to the source-free subspace of the Fock space and then applying to it a particular Murray-von Neumann (MvN) transformation. Such a transformation is not unitary but generalizes the known commutative singular gauge transformation and removes the troublesome source term.[1] The gauge potential producing the projected 't Hooft multi-instanton gauge field cannot be obtained by the standard projection since projecting does not commute with calculating the field strength. The projected configuration may be termed the non-commutative 't Hooft instanton in a singular gauge, but its gauge potential turns out to be given only implicitly. Nevertheless, after the MvN transformation (on the projected field configuration) we obtain the correct non-commutative 't Hooft n-instanton, which contains all known explicit solutions as special cases.

2. Instantons from the twistor approach

Definitions and notation. We consider the euclidean space $\mathbb{R}^4$ with the metric $\delta_{\mu\nu}$, a gauge potential $A = A_\mu \mathrm{d}x^\mu$ and the Yang-Mills field $F = \mathrm{d}A + A \wedge A$ with components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\partial_\mu := \partial/\partial x^\mu$ and $\mu, \nu, \ldots = 1, 2, 3, 4$. The fields A_μ and $F_{\mu\nu}$ take values in the Lie algebra $u(2)$.

The self-dual Yang-Mills (SDYM) equations have the form:

$$*\, F = F \Longrightarrow \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = F_{\mu\nu}\,, \tag{2.1}$$

where $*$ denotes the Hodge star operator and $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in $\mathbb{R}^4$, with $\varepsilon_{1234} = 1$. Solutions of (2.1) having finite Yang-Mills action are called

[1] For the one-instanton configuration such a transformation was considered by Furuuchi [9] in the ADHM approach.

instantons. Their action

$$S = -\frac{1}{g^2}\int \mathrm{tr}\ F \wedge *F \tag{2.2}$$

equals $8\pi^2/g^2$ times an integer which is the topological charge

$$Q = -\frac{1}{8\pi^2}\int \mathrm{tr}\ F \wedge F\,. \tag{2.3}$$

By "tr" we denote the trace over the $u(2)$ gauge algebra and by g the Yang-Mills coupling constant hidden in the definition of the Lie algebra components of the fields A and F.

If we introduce complex coordinates

$$y = x^1 + \mathrm{i}x^2\,, \qquad z = x^3 - \mathrm{i}x^4\,, \qquad \bar{y} = x^1 - \mathrm{i}x^2\,, \qquad \bar{z} = x^3 + \mathrm{i}x^4 \tag{2.4}$$

and put

$$\begin{aligned} A_y &= \frac{1}{2}(A_1 - \mathrm{i}A_2)\,, \qquad A_z = \frac{1}{2}(A_3 + \mathrm{i}A_4)\,, \\ A_{\bar{y}} &= \frac{1}{2}(A_1 + \mathrm{i}A_2)\,, \qquad A_{\bar{z}} = \frac{1}{2}(A_3 - \mathrm{i}A_4)\,, \end{aligned} \tag{2.5}$$

then the SDYM equations (2.1) will read

$$[D_y, D_z] = 0\,, \qquad [D_{\bar{y}}, D_{\bar{z}}] = 0\,, \qquad [D_y, D_{\bar{y}}] + [D_z, D_{\bar{z}}] = 0\,, \tag{2.6}$$

where $D_\mu := \partial_\mu + A_\mu$. These equations can be obtained as the compatibility condition of the following linear system of equations:

$$(D_{\bar{y}} - \lambda D_z)\,\psi(x,\lambda) = 0 \qquad \text{and} \qquad (D_{\bar{z}} + \lambda D_y)\,\psi(x,\lambda) = 0\,, \tag{2.7}$$

where the 2×2 matrix ψ depends on $(y, \bar{y}, z, \bar{z}, \lambda)$ but not on $\bar{\lambda}$. The "spectral parameter" λ lies in the extended complex plane $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$.

Twistors and transition functions. In fact, the function ψ in (2.7) is defined on the *twistor space* $\mathcal{P} = \mathbb{R}^4 \times \mathbb{C}P^1$ for the space $\mathbb{R}^4$ [39, 40]. The sphere S^2, considered as the complex projective line $\mathbb{C}P^1$, can be covered by two coordinate patches U_+ and U_- with

$$\mathbb{C}P^1 = U_+ \cup U_-\,, \qquad U_+ = \mathbb{C}P^1 \setminus \{\infty\}\,, \qquad U_- = \mathbb{C}P^1 \setminus \{0\}\,, \tag{2.8}$$

and coordinates λ and $\tilde{\lambda}$ on U_+ and U_-, respectively. Therefore, also $\mathcal{P}$ can be covered by two coordinate patches,

$$\mathcal{P} = \mathcal{U}_+ \cup \mathcal{U}_-\,, \qquad \mathcal{U}_+ = \mathbb{R}^4 \times U_+\,, \qquad \mathcal{U}_- = \mathbb{R}^4 \times U_-\,, \tag{2.9}$$

with complex coordinates

$$\begin{aligned} w_1 &= y - \lambda\bar{z}\,, \qquad w_2 = z + \lambda\bar{y}\,, \qquad w_3 = \lambda\,, \qquad \text{and} \\ \tilde{w}_1 &= \tilde{\lambda}y - \bar{z}\,, \qquad \tilde{w}_2 = \tilde{\lambda}z + \bar{y}\,, \qquad \tilde{w}_3 = \tilde{\lambda} \end{aligned} \tag{2.10}$$

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on $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_- \simeq \mathbb{R}^4 \times \mathbb{C}^*$ these coordinates are related by

$$\tilde{w}_1 = \frac{w_1}{w_3}\,, \qquad \tilde{w}_2 = \frac{w_2}{w_3}\,, \qquad \text{and} \qquad \tilde{w}_3 = \frac{1}{w_3}\,. \tag{2.11}$$

On the open set $\mathcal{U}_+ \cap \mathcal{U}_-$ one may use any of them, and we will use w_1, w_2, and $w_3{=}\lambda$.

There exist two matrix-valued solutions $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ of (2.7) which are defined on $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. Finding them, one can introduce the matrix-valued function

$$f_{+-} := \psi_+^{-1}\,\psi_- \tag{2.12}$$

defined on the open set $\mathcal{U}_+ \cap \mathcal{U}_- \subset \mathcal{P}$. From (2.7) it follows that f_{+-} depends on the complex coordinates w_1, w_2, and λ holomorphically,

$$(\partial_{\bar{y}} - \lambda\partial_z) f_{+-} = 0\,, \qquad \text{and} \qquad (\partial_{\bar{z}} + \lambda\partial_y) f_{+-} = 0 \Longrightarrow f_{+-} = f_{+-}(w_1, w_2, \lambda)\,. \tag{2.13}$$

Any such function defines a holomorphic bundle over $\mathcal{P}$. Namely, f_{+-} can be identified with a transition function in a holomorphic bundle over $\mathcal{P}$, and a pair of functions $\psi_\pm$ defines a smooth trivialization of this bundle.

Gauge equivalence and reality conditions. It is easy to see that gauge transformations

$$A_\mu \longmapsto A_\mu^g = g^{-1} A_\mu\, g + g^{-1}\partial_\mu\, g \tag{2.14}$$

are induced by the transformations

$$\psi_+ \longmapsto \psi_+^g = g^{-1}\psi_+ \qquad \text{and} \qquad \psi_- \longmapsto \psi_-^g = g^{-1}\psi_-\,, \tag{2.15}$$

where $g{=}g(x)$ is an arbitrary U(2)-valued function on $\mathbb{R}^4$. The transition function $f_{+-} = \psi_+^{-1}\psi_-$ is invariant under these transformations. On the other hand, the gauge potential A is inert under the transformations

$$\psi_+ \longmapsto \psi_+\, h_+^{-1} \qquad \text{and} \qquad \psi_- \longmapsto \psi_-\, h_-^{-1}\,, \tag{2.16}$$

where $h_+{=}h_+(w_1, w_2, \lambda)$ and $h_-{=}h_-(\tilde{w}_1, \tilde{w}_2, \tilde{\lambda})$ are arbitrary matrix-valued holomorphic functions on $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively.

The reality of the gauge fields is an important issue [39, 40]. The antihermiticity conditions $A_\mu^\dagger = -A_\mu$ for components of the gauge potential imply the following "reality" conditions for the matrices $\psi_\pm$ and f_{+-}:

$$\psi_+^\dagger(x, -\bar{\lambda}^{-1}) = \psi_-^{-1}(x,\lambda) \qquad \text{and} \qquad f_{+-}^\dagger(x, -\bar{\lambda}^{-1}) = f_{+-}(x,\lambda)\,. \tag{2.17}$$

Splitting of transition functions. Consider now the inverse situation. Let us have a holomorphic matrix-valued function f_{+-} on the open subset $\mathcal{U}_+ \cap \mathcal{U}_-$ of the twistor space $\mathcal{P}$. Suppose we are able to split f_{+-}, i.e. for each fixed $x \in \mathbb{R}^4$ find matrix-valued functions $\psi_\pm(x,\lambda)$ such that $f_{+-} = \psi_+^{-1}\psi_-$ on $U_+ \cap U_-$ and the functions ψ_+ and ψ_- can be extended continuously to functions regular on U_+ and U_-, respectively. From the holomorphicity of f_{+-} it then follows that

$$\begin{aligned}\psi_+(\partial_{\bar y}-\lambda\partial_z)\psi_+^{-1} &= \psi_-(\partial_{\bar y}-\lambda\partial_z)\psi_-^{-1} \qquad \text{and}\\ \psi_+(\partial_{\bar z}+\lambda\partial_y)\psi_+^{-1} &= \psi_-(\partial_{\bar z}+\lambda\partial_y)\psi_-^{-1}\,.\end{aligned} \tag{2.18}$$

Recall that the matrix-valued functions ψ_+ and ψ_- are regular on their respective domains, so that we may expand them into power series in λ and λ^{-1}, respectively, $\psi_\pm = \sum_{n\geq 0}\lambda^{\pm n}\psi_{\pm,n}(x)$. Upon substituting into (2.18) one easily sees that both sides of (2.18) must be linear in λ, and one can introduce A_μ by

$$A_{\bar y}-\lambda A_z = \psi_\pm(\partial_{\bar y}-\lambda\partial_z)\psi_\pm^{-1} \qquad \text{and} \qquad A_{\bar z}+\lambda A_y = \psi_\pm(\partial_{\bar z}+\lambda\partial_y)\psi_\pm^{-1}\,. \tag{2.19}$$

Hence, the gauge field components may be calculated from

$$A_{\bar y} = \psi_+\partial_{\bar y}\psi_+^{-1}|_{\lambda=0} = -A_y^\dagger \qquad \text{and} \qquad A_{\bar z} = \psi_+\partial_{\bar z}\psi_+^{-1}|_{\lambda=0} = -A_z^\dagger\,. \tag{2.20}$$

By construction, the components $\{A_\mu\}$ of the gauge potential A defined by (2.19) or (2.20) satisfy the SDYM equations. For more detailed discussion of *local* solutions, their infinite-dimensional moduli space and references see e.g. [41, 42].

For a fixed point $x\in\mathbb{R}^4$, the task to split a matrix-valued holomorphic function $f_{+-}\equiv f_{+-}(y-\lambda\bar z, z+\lambda\bar y,\lambda)$ defines a *parametric Riemann-Hilbert problem* on $\mathbb{C}P^1$. The explicit general solution of this Riemann-Hilbert problem is not known. For a large class of special cases, however, the splitting can be achieved. In this paper we shall make explicit use of a particular example, the so-called Atiyah-Ward ansatz [39], to be presented momentarily in the non-commutative context.

3. The Atiyah-Ward ansatz

Non-commutative Yang-Mills theory. The non-commutative deformation of (classical) field theory is most easily effected by extending the function product in field space to the star product

$$(f\star g)(x) = f(x)\,\exp\left\{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu\right\}g(x)\,, \tag{3.1}$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. In this work, we restrict ourselves to the case of a self-dual ($\epsilon=1$) or an anti-self-dual ($\epsilon=-1$) tensor $\theta^{\mu\nu}$ and choose coordinates such that

$$\theta^{12} = -\theta^{21} = \epsilon\theta^{34} = -\epsilon\theta^{43} = \theta > 0\,. \tag{3.2}$$

In star-product formulation, the SDYM equations (2.1) are formally unchanged, but the components of the non-commutative field strength now read

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu\star A_\nu - A_\nu\star A_\mu\,. \tag{3.3}$$

The nonlocality of the star product renders explicit computations cumbersome. We therefore take advantage of the Moyal-Weyl correspondence and pass to the operator formalism, which trades the star product for operator-valued coordinates $\hat x^\mu$ satisfying

$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}$. This defines the non-commutative euclidean space $\mathbb{R}^4_\theta$. In complex coordinates (2.4) our choice (3.2) implies

$$[\,\hat{y}\,,\,\hat{\bar{y}}\,] = 2\theta\,, \qquad [\,\hat{z}\,,\,\hat{\bar{z}}\,] = -2\epsilon\theta\,, \qquad \text{and other commutators} = 0\,. \tag{3.4}$$

Clearly, coordinate derivatives are now inner derivations of this algebra, i.e.

$$\hat{\partial}_y \hat{f} = \frac{-1}{2\theta}\,[\,\hat{\bar{y}}\,,\,\hat{f}\,] \qquad \text{and} \qquad \hat{\partial}_{\bar{y}} \hat{f} = \frac{1}{2\theta}\,[\,\hat{y}\,,\,\hat{f}\,] \tag{3.5}$$

for any function $\hat{f}$ of $(\hat{y},\hat{\bar{y}},\hat{z},\hat{\bar{z}})$. Analogous formulae hold for $\hat{\partial}_z$ and $\hat{\partial}_{\bar{z}}$.

The obvious representation space for the Heisenberg algebra (3.4) is the two-oscillator Fock space $\mathcal{H}$ spanned by $\{|n_1,n_2\rangle$ with $n_1,n_2 = 0,1,2,\ldots\}$. In $\mathcal{H}$ one can introduce an integer ordering of states e.g. as follows [37]:

$$|k\rangle = |n_1,n_2\rangle = \frac{\hat{\bar{y}}^{n_1}\,\hat{\bar{z}}_\epsilon^{\,n_2}\,|0,0\rangle}{\sqrt{n_1!n_2!(2\theta)^{n_1+n_2}}}\,, \qquad \text{with} \;\; \hat{z}_\epsilon := \frac{1-\epsilon}{2}\,\hat{z} + \frac{1+\epsilon}{2}\,\hat{\bar{z}} \tag{3.6}$$

and $k = n_1+\frac{1}{2}(n_1+n_2)(n_1+n_2+1)$. Thus, coordinates as well as fields are to be regarded as operators in $\mathcal{H}$. The Moyal-Weyl map yields the operator equivalent of star multiplication and integration,

$$f \star g \;\longmapsto\; \hat{f}\,\hat{g} \qquad \text{and} \qquad \int \mathrm{d}^4x\, f = (2\pi\theta)^2\,\mathrm{Tr}_{\mathcal{H}}\,\hat{f}\,, \tag{3.7}$$

respectively, where "$\mathrm{Tr}_{\mathcal{H}}$" signifies the trace over the Fock space $\mathcal{H}$.

In the operator formulation, the non-commutative generalization of the SDYM equations (2.6) again retains their familiar form,

$$\hat{F}_{yz} = 0\,, \qquad \hat{F}_{\bar{y}\bar{z}} = 0\,, \qquad \hat{F}_{y\bar{y}} + \hat{F}_{z\bar{z}} = 0\,. \tag{3.8}$$

The operator-valued field-strength components $\hat{F}_{\mu\nu}$, however, now relate to the non-commutative gauge-potential components $\hat{A}_\mu$ with the help of (3.5), as e.g. in

$$2\theta\,\hat{F}_{yz} = [-\hat{\bar{y}} + \theta\hat{A}_y, \hat{A}_z] - [\epsilon\hat{\bar{z}} + \theta\hat{A}_z, \hat{A}_y]\,. \tag{3.9}$$

For the rest of the paper we shall work in the operator formalism and drop the hats over the operators in order to avoid cluttering the notation.

Non-commutative Atiyah-Ward ansatz. Commutative instantons can be obtained by the famous ADHM construction [5], which was derived from the twistor approach. Almost all works on non-commutative instantons are based on the modified ADHM construction [4]. At the same time, it is known that the modified ADHM construction can be interpreted in terms of a non-commutative version of the twistor transform [33, 34]. Therefore it is reasonable to expect that an approach based on the splitting of transition functions in a holomorphic bundle over a non-commutative twistor space [33, 34, 38] will work as well. Here we show that this is indeed the case for the simplest Atiyah-Ward ansatz for the transition functions.

In the commutative situation the infinite hierarchy of Atiyah-Ward ansätze reads

$$f_{+-}^{(k)} = \begin{pmatrix} \lambda^k & 0 \\ \rho & \lambda^{-k} \end{pmatrix}, \tag{3.10}$$

where $k = 1, 2, \ldots$ and ρ denotes a holomorphic *function* on $\mathcal{U}_+ \cap \mathcal{U}_- \subset \mathcal{P}$.[2] We are confident that the ansätze (3.10) allow one to construct solutions of the non-commutative SDYM equations (3.8) simply by promoting ρ to an *operator* acting in the Fock space $\mathcal{H}$.

Specializing to $k = 1$, we introduce the matrix

$$f_{+-} := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} f_{+-}^{(1)} = \begin{pmatrix} \rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}. \tag{3.11}$$

Because f_{+-} is related to $f_{+-}^{(1)}$ by a transformation (2.16), with $h_+ = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $h_- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it leads to the same gauge field configuration. Yet, f_{+-} has the advantage of satisfying the reality condition (2.17). Here ρ is a holomorphic "real" operator-valued function, i.e.

$$(\partial_{\bar{y}} - \lambda\partial_z)\rho = (\partial_{\bar{z}} + \lambda\partial_y)\rho = 0 \qquad \text{and} \qquad \rho^\dagger(x, -\bar{\lambda}^{-1}) = \rho(x, \lambda)\,. \tag{3.12}$$

It is useful to expand ρ in a Laurent series in λ,

$$\rho \;=\; \sum_{m=-\infty}^{\infty} \rho_m\,\lambda^m = \rho_- + \phi + \rho_+\,, \tag{3.13}$$

where

$$\rho_+ := \sum_{m>0} \rho_m\,\lambda^m\,, \qquad \rho_- := \sum_{m<0} \rho_m\,\lambda^m\,, \qquad \text{and} \qquad \phi := \rho_0\,. \tag{3.14}$$

The reality condition (2.17) then becomes

$$\phi^\dagger(x) = \phi(x) \qquad \text{and} \qquad \rho_+^\dagger(x, -\bar{\lambda}^{-1}) = \rho_-(x, \lambda)\,. \tag{3.15}$$

It is not difficult to see that f_{+-} can be split as $f_{+-} = \psi_+^{-1}\psi_-$, where

$$\psi_- = \frac{1}{\sqrt{\phi}}\begin{pmatrix} \phi + \rho_- & \lambda^{-1} \\ \lambda\rho_- & 1 \end{pmatrix} \qquad \text{and} \qquad \psi_+^{-1} = \begin{pmatrix} \phi + \rho_+ & -\lambda^{-1}\rho_+ \\ -\lambda & 1 \end{pmatrix}\frac{1}{\sqrt{\phi}}\,. \tag{3.16}$$

Consequently we have

$$\psi_-^{-1} = \begin{pmatrix} 1 & -\lambda^{-1} \\ -\lambda\rho_- & \phi + \rho_- \end{pmatrix}\frac{1}{\sqrt{\phi}} \qquad \text{and} \qquad \psi_+ = \frac{1}{\sqrt{\phi}}\begin{pmatrix} 1 & \lambda^{-1}\rho_+ \\ \lambda & \phi + \rho_+ \end{pmatrix}, \tag{3.17}$$

which satisfy the reality condition (2.17). Thus, the first Atiyah-Ward ansatz is easily generalized to the non-commutative case.

[2] A reminder on the commutative situation: for k=1 the ansatz (3.10) leads to a parametrization of self-dual gauge fields in terms of a scalar field $\phi = \mathrm{res}_{\lambda=0}(\lambda^{-1}\rho)$ satisfying the wave equation, while for $k \geq 2$ it produces solutions more general than the 't Hooft n-instanton configurations [39, 43, 44, 45]. Note that the matrices $f_{+-}^{(k)}$ do not satisfy the reality condition (2.17).

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Parametrization of the gauge potential. From the operator version of formulae (2.19)–(2.20) and recursion relations

$$\partial_{\bar y}\,\rho_{m+1} = \partial_z \rho_m \qquad \text{and} \qquad \partial_{\bar z}\,\rho_{m+1} = -\partial_y \rho_m \tag{3.18}$$

implied by (3.12) for the Laurent coefficients in (3.13) we get

$$\begin{aligned}
A_{\bar y} &= \begin{pmatrix} \phi^{-1/2}\partial_{\bar y}\phi^{1/2} & -\phi^{-1/2}(\partial_z\phi)\phi^{-1/2} \\ 0 & \phi^{1/2}\partial_{\bar y}\phi^{-1/2} \end{pmatrix}, \\
A_{\bar z} &= \begin{pmatrix} \phi^{-1/2}\partial_{\bar z}\phi^{1/2} & \phi^{-1/2}(\partial_y\phi)\phi^{-1/2} \\ 0 & \phi^{1/2}\partial_{\bar z}\phi^{-1/2} \end{pmatrix}, \\
A_{y} &= \begin{pmatrix} \phi^{1/2}\partial_{y}\phi^{-1/2} & 0 \\ \phi^{-1/2}(\partial_{\bar z}\phi)\phi^{-1/2} & \phi^{-1/2}\partial_{y}\phi^{1/2} \end{pmatrix}, \\
A_{z} &= \begin{pmatrix} \phi^{1/2}\partial_{z}\phi^{-1/2} & 0 \\ -\phi^{-1/2}(\partial_{\bar y}\phi)\phi^{-1/2} & \phi^{-1/2}\partial_{z}\phi^{1/2} \end{pmatrix}.
\end{aligned} \tag{3.19}$$

Rewriting these expressions in real coordinates x^μ, we obtain the non-commutative generalization of the CFtHW ansatz,

$$A_\mu = \bar\eta^a_{\mu\nu}\frac{\sigma_a}{2\mathrm{i}}\left(\phi^{1/2}\partial_\nu\phi^{-1/2} - \phi^{-1/2}\partial_\nu\phi^{1/2}\right) + \frac{\mathbf{1}_2}{2}\left(\phi^{-1/2}\partial_\mu\phi^{1/2} + \phi^{1/2}\partial_\mu\phi^{-1/2}\right), \tag{3.20}$$

where

$$\bar\eta^a_{\mu\nu} = \begin{cases} \epsilon^a_{bc}\,, & \text{for } \mu=b\,,\ \nu=c\,, \\ -\delta^a_\mu\,, & \text{for } \nu=4\,, \\ \delta^a_\nu\,, & \text{for } \mu=4 \end{cases} \tag{3.21}$$

is the anti-self-dual 't Hooft tensor [44], and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.22}$$

are the Pauli matrices.

Reduced SDYM equation. Calculating the Yang-Mills curvature for the non-commutative CFtHW ansatz (3.20) results in

$$F_{\bar y\bar z} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \qquad F_{yz} = \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix}, \qquad F_{y\bar y} + F_{z\bar z} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \tag{3.23}$$

with

$$X = \phi^{-1/2}(\partial_y\partial_{\bar y}\phi + \partial_z\partial_{\bar z}\phi)\,\phi^{-1/2}\,. \tag{3.24}$$

Hence, for the ansatz (3.20) the non-commutative SDYM equations (3.8) are reduced to

$$\phi^{-1/2}(\partial_y\partial_{\bar y}\phi + \partial_z\partial_{\bar z}\phi)\,\phi^{-1/2} = 0\,. \tag{3.25}$$

It is natural to assume [4] that a solution ϕ_n of this equation looks exactly like the standard 't Hooft solution

$$\phi_n = 1 + \sum_{i=1}^n \frac{\Lambda_i^2}{(x^\mu - b_i^\mu)(x^\mu - b_i^\mu)} =: 1 + \sum_{i=1}^n \frac{\Lambda_i^2}{r_i^2}\,, \tag{3.26}$$

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where now the x^μ are operators and

$$\begin{aligned}
\frac{1}{x^\mu x^\mu} &:= \frac{1}{2\theta} \sum_{n_1,n_2\geq 0} \frac{1}{n_1{+}n_2{+}1}\, |n_1,n_2\rangle\langle n_1,n_2|\,, \\
r_i^2 &= \delta_{\mu\nu}(x^\mu - b_i^\mu)(x^\nu - b_i^\nu) = \bar{y}_i y_i + z_{i\epsilon}\bar{z}_{i\epsilon} = \bar{y}_i y_i + \bar{z}_{i\epsilon} z_{i\epsilon} + 2\theta\,, \\
y_i &:= y - b_i^y\,, \quad z_{i\epsilon} := z_\epsilon - b_i^z\,, \\
b_i &:= (b_i^\mu) = (b_i^y, b_i^z)\,, \qquad \bar{b}_i := (\bar{b}_i^y, \bar{b}_i^z)\,.
\end{aligned} \tag{3.27}$$

The real parameters b_i^μ and Λ_i denote the position coordinates and the scale of the ith instanton. In particular, as a candidate for the one-instanton solution we have (putting $b_1^\mu{=}0$)

$$\phi_1 = 1 + \frac{\Lambda^2}{r^2}\,, \tag{3.28}$$

where

$$r^2 = \delta_{\mu\nu} x^\mu x^\nu = \frac{1}{2}(\bar{y}y + y\bar{y} + \bar{z}z + z\bar{z}) = \bar{y}y + z_\epsilon \bar{z}_\epsilon = \bar{y}y + \bar{z}_\epsilon z_\epsilon + 2\theta\,. \tag{3.29}$$

Sources. When checking the self-duality of the purported one-instanton configuration one discovers a subtlety: the substitution of (3.28) into the reduced SDYM equation (3.25) and results [46] on Green's functions reveal that the deviation from self-duality,

$$\begin{aligned}
X_1 := \phi_1^{-1/2}(\partial_y\partial_{\bar{y}}\phi_1 + \partial_z\partial_{\bar{z}}\phi_1)\,\phi_1^{-1/2} &= -\frac{\Lambda^2}{2\theta(\Lambda^2+2\theta)}\,P_0\,, \\
\text{with } P_0 := |0,0\rangle\langle 0,0|\,,&
\end{aligned} \tag{3.30}$$

is not zero if $\Lambda^2{\neq}0$. This has led the authors of [18] to the conclusion that this configuration is *not* self-dual. This is not the whole story — one can show that all ϕ_n in (3.26) fail to satisfy (3.25). Namely, by differentiating we obtain

$$\partial_y\partial_{\bar{y}}\phi_n + \partial_z\partial_{\bar{z}}\phi_n = -\frac{1}{4\theta^2}\sum_{i=1}^n \Lambda_i^2\, |b_i\rangle\langle b_i|\,, \tag{3.31}$$

where

$$|b_i\rangle := |b_i^y, b_i^z\rangle = \mathrm{e}^{-\frac{1}{2}|b_i|^2}\,\mathrm{e}^{b_i^y \bar{y}/\sqrt{2\theta}}\,\mathrm{e}^{b_i^z \bar{z}_\epsilon/\sqrt{2\theta}}\,|0,0\rangle \tag{3.32}$$

denotes the "shifted ground state" centered at (b_i^μ) which is constructed as a coherent state for the Heisenberg-Weyl group generated by the algebra (3.4). The vectors $|b_i\rangle$, $i = 1,\dots,n$, span an n-dimensional subspace of $\mathcal{H}$. Since they are not orthonormalized it is useful to also introduce an orthonormal basis $\{|h_1\rangle,\dots,|h_n\rangle\}$ of this subspace through [47]

$$(\,|h_1\rangle,\, |h_2\rangle, \dots,\, |h_n\rangle\,) := T\,(T^\dagger T)^{-1/2}\,, \qquad \text{with} \qquad T := (\,|b_1\rangle,\, |b_2\rangle, \dots,\, |b_n\rangle\,)\,. \tag{3.33}$$

This basis can be extended to an orthonormal basis of the whole Fock space $\mathcal{H}$ by simply adjoining further vectors $|h_{n+1}\rangle, |h_{n+2}\rangle, \dots$. From the explicit form of the function ϕ_n it follows that the matrix elements $\langle h_i|\phi_n^{-1/2}|h_j\rangle$ are nonzero only for $i,j \leq n$. Therefore, the operator $\phi_n^{-1/2}$ preserves the subspace of $\mathcal{H}$ spanned by the vectors $|b_i\rangle$, $i = 1,\dots,n$, i.e.

$$\phi_n^{-1/2}|b_i\rangle = \sum_{j=1}^{n} M_{ij}\,|b_j\rangle \qquad \text{and} \qquad \langle b_i|\,\phi_n^{-1/2} = \sum_{k=1}^{n}\langle b_k|\,\overline{M}_{ki}\,, \tag{3.34}$$

where $M_{ij} = M_{ij}(\theta,\Lambda_1^2,\dots,\Lambda_n^2)$ are some constants. Using this fact we arrive at

$$X_n := \phi_n^{-1/2}(\partial_y\partial_{\bar{y}}\phi_n + \partial_z\partial_{\bar{z}}\phi_n)\,\phi_n^{-1/2} = -\frac{1}{4\theta^2}\sum_{i,j,k=1}^{n}|b_j\rangle\,\overline{M}_{ki}\,\Lambda_i^2\,M_{ij}\,\langle b_k|\,, \tag{3.35}$$

which is not zero if at least one Λ_i^2 does not vanish. Appendix A derives the same expression via the ADHM approach.

4. Non-commutative instantons

Projected field configurations. The twistor approach has given us a systematic way of producing a non-commutative generalization (3.20) of the CFtHW ansatz. However, it has been shown above that the non-commutative 't Hooft type ansatz (3.26) does not solve the SDYM equations since $X_n \neq 0$ produce sources in the r.h.s. of (3.23). These sources are localized on a finite-dimensional subspace of the Fock space.

Let us look at the situation in more detail. Recall that in the non-commutative case the components A_μ and $F_{\mu\nu}$ are *operators* acting (on the left) in the space $\mathcal{H}\otimes\mathbb{C}^2 = \mathcal{H}\oplus\mathcal{H}$ which carries a fundamental representation of the group U(2). It is easy to see that for the ansatz (3.28) each term of the operators A_μ in (3.20) annihilates the state $|0,0\rangle\otimes\mathbb{C}^2$ or $\mathbb{C}^2\otimes\langle 0,0|$ (when acting on the right), thus $\langle 0,0|A_\mu|0,0\rangle = 0$. This shows that the A_μ are well defined only on the subspace $(1{-}P_0)\mathcal{H}\otimes\mathbb{C}^2$ of the Fock space. Moreover, from (3.30) we see that the gauge fields are self-dual only on the same subspace since $X_1(1{-}P_0) = 0$. Analogously, one can easily see that X_n given by (3.35) is annihilated by the projector

$$1 - P_{n-1} := 1 - \sum_{i,j=1}^{n}|b_i\rangle\frac{1}{\langle b_i|b_j\rangle}\langle b_j| = 1 - \sum_{i=1}^{n}|h_i\rangle\langle h_i| \tag{4.1}$$

onto the orthogonal complement of the subspace spanned by $\{|b_i\rangle, i = 1,\dots,n\}$ but not outside it. Therefore, the solutions based on (3.26) are self-dual on the reduced Fock space $(1{-}P_{n-1})\mathcal{H}\otimes\mathbb{C}^2$ (projective Fock module). The same phenomenon occurs in the modified ADHM construction of the non-commutative one- and two-instanton solutions and was discussed intensively by Ho [7] and especially by Furuuchi [8]–[11]. The ADHM construction of non-commutative 't Hooft n-instantons including the appearance of a source term for any n is described in appendix A. Note that the deletion of a subspace generated by n states $|b_i\rangle$, $i = 1,\dots,n$, from the Fock space corresponds to the exclusion of n points b_i, $i = 1,\dots,n$, (in which the gauge potential is singular) from the commutative space $\mathbb{R}^4$. Hence, the sources in the r.h.s. of (3.23), being localized on the subspace $P_{n-1}\mathcal{H}\otimes\mathbb{C}^2$, imply that the ansatz (3.26) produces a *singular* solution.

Quite generally, consider any finite-rank projector P. If the deviation from self-duality is localized on the subspace $P\mathcal{H}\otimes\mathbb{C}^2$ of the Fock space then obviously we can produce a self-dual field strength $F^p_{\mu\nu}$ by projecting the gauge field onto the reduced Fock space $(1-P)\mathcal{H}\otimes\mathbb{C}^2$,

$$F_{\mu\nu} \longmapsto F^p_{\mu\nu} = (1-P)\,F_{\mu\nu}\,(1-P)\,. \tag{4.2}$$

Unfortunately, the corresponding gauge potential A^p_μ generating this $F^p_{\mu\nu}$ *cannot* be obtained by naively projecting A_μ, since projecting does not commute with calculating the field strength:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "?"'] & F \arrow[d, "1-P"] \\ A^p \arrow[r, "\delta_p"] & F^p\,, \end{tikzcd}$$

where $\downarrow_{1-P}$ denotes the standard projection onto $(1-P)\mathcal{H}\otimes\mathbb{C}^2$ and δ and δ_p compute the curvature in the full and the reduced Fock space, respectively. More explicitly, it reads

$$\begin{aligned} F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} + [A_\mu\,,A_\nu]\,, \\ F^p_{\mu\nu} &= (1-P)\,\partial_{[\mu}A^p_{\nu]}\,(1-P) + [A^p_\mu\,,A^p_\nu] + (1-P)\,\partial_{[\mu}P\,\partial_{\nu]}P\,. \end{aligned} \tag{4.3}$$

Although A^p is a projected gauge potential, i.e. it fulfils

$$A^p_\mu = (1-P)\,A^p_\mu\,(1-P)\,, \tag{4.4}$$

a short calculation reveals that it differs from the standard projection of A.[3] To avoid confusion, we call A^p_μ the "projected" gauge potential. As the question mark in the above diagram indicates, it is given only *implicitly* as a function of A_μ through

$$(1-P)\Big(\partial_{[\mu}A_{\nu]} + [A_\mu\,,A_\nu]\Big)(1-P) = (1-P)\Big(\partial_{[\mu}A^p_{\nu]} + [A^p_\mu\,,A^p_\nu] + \partial_{[\mu}P\,\partial_{\nu]}P\Big)(1-P)\,. \tag{4.5}$$

Similarly to the question of deriving some gauge potential A from a given field strength F in ordinary Yang-Mills theory, it seems difficult to assert the existence of A^p solving (4.5) for a given A. Our analogue of the Bianchi identity are the necessary conditions[4]

$$(1-P)\,\partial_{[\mu}F^p_{\nu\rho]}P = -F^p_{[\mu\nu}\partial_{\rho]}P \qquad \text{and} \qquad (1-P)\,\partial_{[\mu}F^p_{\nu\rho]}(1-P) = [F^p_{[\mu\nu}\,,A^p_{\rho]}]\,, \tag{4.6}$$

of which the first one is solved by (4.2) but the second one is nontrivial. Nevertheless, from the existence of the canonical solution $A^p{=}A$ for $\theta{=}0$ and the smoothness of all expressions in the deformation parameter θ we derive some confidence that A^p should exist at least for sufficiently small values of θ.

[3] In a previous version of the paper we erroneously identified A^p with $(1-P)A(1-P)$. We thank Diego Correa and Fidel Schaposnik for questions which exposed this confusion to us.

[4] O.L. thanks Martin Roček for a discussion on this point.

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Murray-von Neumann transformations. After projecting our gauge field $F \mapsto F^p$ via (4.2) and finding a solution A^p of (4.5) we obtain a self-dual configuration (A^p, F^p) which may be termed the non-commutative 't Hooft multi-instanton (A^p, F^p) *in a singular gauge*. As is well known in the commutative situation (see e.g. [48]), the singularity in the gauge potential can be removed by a singular gauge transformation, leading for instance to the BPST form of the one-instanton solution. The non-commutative analogues of such singular gauge transformations are so-called Murray-von Neumann (MvN) transformations. Indeed, such transformations were proposed in a different context — the modified ADHM approach — to remove the singularity of the non-commutative one-instanton solution for the U(1) [7] and U(2) [9] gauge groups. Here we will show that also for the non-commutative *'t Hooft solutions* there exist MvN transformations which, in combination with suitable projections, repair the deficiency in (3.35) and produce *regular* solutions for any finite n.

Given our finite-rank projector P, we consider a special kind of MvN transformations (partial isometry [49]),

$$A^p_\mu \longmapsto A'_\mu = V^\dagger A^p_\mu V + V^\dagger \partial_\mu V \qquad \text{and} \qquad F^p_{\mu\nu} \longmapsto F'_{\mu\nu} = V^\dagger F^p_{\mu\nu} V\,, \tag{4.7}$$

where the intertwining operators

$$V:\ \mathcal{H}\otimes\mathbb{C}^2 \longrightarrow (1-P)\mathcal{H}\otimes\mathbb{C}^2 \qquad \text{and} \qquad V^\dagger:\ (1-P)\mathcal{H}\otimes\mathbb{C}^2 \longrightarrow \mathcal{H}\otimes\mathbb{C}^2 \tag{4.8}$$

satisfy the relation

$$V^\dagger V = 1\,, \qquad \text{while} \qquad V V^\dagger = 1 - P \tag{4.9}$$

upon extension to $\mathcal{H}\otimes\mathbb{C}^2$. The configuration (A^p, F^p) lives on the projective Fock module $(1-P)\mathcal{H}\otimes\mathbb{C}^2$, but (A', F') exists on the free Fock module $\mathcal{H}\otimes\mathbb{C}^2$. Therefore, the gauge fields are related to their potentials by (4.3) and

$$F'_{\mu\nu} = \partial_{[\mu}A'_{\nu]} + [A'_\mu\,, A'_\nu]\,. \tag{4.10}$$

Via the natural embedding of the reduced Fock space into the full Fock space, we may extend V and $V^\dagger$ to endomorphisms of $\mathcal{H}\otimes\mathbb{C}^2$ by declaring $V^\dagger P = 0 = VP$. Although the extended V is not unitary, the transformations (4.7) can be regarded as a non-commutative version of singular gauge transformations. Furthermore, we may actually bypass the projection as far as the field strength is concerned and directly compute

$$F'_{\mu\nu} = V^\dagger F_{\mu\nu} V\,. \tag{4.11}$$

In the following subsections we will demonstrate that to the singular configuration (3.20) with ϕ from (3.26) one should apply firstly a projection onto the reduced Fock space $(1-P_{n-1})\mathcal{H}\otimes\mathbb{C}^2$ to obtain the non-commutative 't Hooft multi-instanton *in a singular gauge*, and secondly an MvN transformation to arrive at the *non-singular* non-commutative 't Hooft n-instanton solution. In short,

$$(A_\mu, F_{\mu\nu}) \overset{1-P_{n-1}}{\longmapsto} (A^p_\mu, F^p_{\mu\nu}) \overset{V_n}{\longmapsto} (A'_\mu, F'_{\mu\nu}) \tag{4.12}$$

will bring us to a satisfactory instanton configuration. However, we will not be able to present explicit expressions for A^p or A'.

One-instanton solution. Let us consider first the one-instanton configuration (3.28) in (3.20), project it onto $(1{-}P_0)\mathcal{H}\otimes\mathbb{C}^2$, and construct its MvN transformation matrix $V{\equiv}V_1$. Due to the relation $(1{-}P_0)X_1(1{-}P_0)=0$, the projected field strength

$$F^p_{\mu\nu} = (1{-}P_0)\, F_{\mu\nu}\,(1{-}P_0) \tag{4.13}$$

is obviously self-dual, but it is singular because ill-defined on $|0,0\rangle\otimes\mathbb{C}^2$. As already mentioned, we cannot rigorously prove the existence of the "projected" gauge potential A^p via (4.5) but we shall assume it henceforth.

As will be justified below, the subsequent MvN transformation applied to (A^p,F^p) is conveniently factorized as $V_1=\check{S}_1\,U_1$, with

$$\check{S}_1 = \begin{pmatrix} 1 & 0 \\ 0 & S_1 \end{pmatrix} \qquad \text{and} \qquad U_1 = \begin{pmatrix} \bar{z}_\epsilon & \bar{y} \\ y & -z_\epsilon \end{pmatrix} \frac{\mathrm{i}}{r}\,, \tag{4.14}$$

where

$$S_1^\dagger\, S_1 = 1\,, \qquad \text{while} \qquad S_1\, S_1^\dagger = 1-P_0\,. \tag{4.15}$$

These relations are satisfied by the shift operator

$$S_1 = \sum_{k\geq 0} |k{+}1\rangle\langle k|\,, \tag{4.16}$$

constructed from the integer ordered states (3.6). Another possible realization of S_1 is [11]

$$S_1 = 1 + \sum_{n_2\geq 0} \big(|0,n_2{+}1\rangle\langle 0,n_2| - |0,n_2\rangle\langle 0,n_2|\big)\,. \tag{4.17}$$

A third possible choice (which we shall use) is a "mixture" of (4.16) and (4.17). Namely, recall that the index k in (3.6) introduces an integer ordering of states in the two-oscillator Fock space $\mathcal{H}$:

$$\{|k\rangle\} = \{|0,0\rangle, |0,1\rangle, |1,0\rangle, |0,2\rangle, \ldots\}\,. \tag{4.18}$$

For fixed instanton number $n{>}1$ let us move the states $|0,i\rangle$ with $0<i\leq n{-}1$ in the sequence (4.18) to the left and enumerate this new order by $\hat{k}$,

$$\{|\hat{k}\rangle\} = \{|0,0\rangle, |0,1\rangle, \ldots, |0,n{-}1\rangle, |1,0\rangle, \ldots\}\,. \tag{4.19}$$

Then, the shift operator

$$S_1 = \sum_{k\geq 0} |\hat{k}{+}1\rangle\langle\hat{k}| \tag{4.20}$$

with the new ordering of states will also satisfy (4.15).

We will now confirm our claim that

$$A'_\mu = V_1^\dagger A^p_\mu V_1 + V_1^\dagger\partial_\mu V_1\,, \qquad \text{and} \qquad F'_{\mu\nu} = V_1^\dagger F^p_{\mu\nu} V_1 \;\equiv\; V_1^\dagger F_{\mu\nu} V_1 \tag{4.21}$$

brings us to the non-singular non-commutative one-instanton solution.[5] With the help of

[5]It is interesting to note that $\check{S}_1^\dagger A^p \check{S}_1 + \check{S}_1^\dagger \mathrm{d}\check{S}_1$ corresponds to the (singular) U(2) one-instanton configuration produced by the modified ADHM construction [9]. It still fails to be self-dual on $P_0\mathcal{H}$.

the definitions (cf. [9])

$$r_0 := (r^2 - 2\theta)^{1/2} = (\bar{y}y + \bar{z}_\epsilon z_\epsilon)^{1/2}\,, \qquad \text{and}$$
$$r_0^{-1} := (1{-}P_0)(\bar{y}y + \bar{z}_\epsilon z_\epsilon)^{-1/2}(1{-}P_0) = \frac{1}{\sqrt{2\theta}} \sum_{n_1,n_2\neq 0} \frac{|n_1 n_2\rangle\langle n_1 n_2|}{\sqrt{n_1{+}n_2}} \tag{4.22}$$

it is readily checked that

$$V_1^\dagger\, V_1 = 1 \otimes \mathbf{1}_2 \qquad \text{but} \qquad V_1\, V_1^\dagger = \begin{pmatrix} 1{-}P_0 & 0 \\ 0 & 1{-}P_0 \end{pmatrix} = (1 - P_0) \otimes \mathbf{1}_2\,. \tag{4.23}$$

Furthermore, one finds that

$$X_1\, V_1 = -\frac{\Lambda^2}{2\theta(\Lambda^2 + 2\theta)} P_0\, V_1 = 0\,, \tag{4.24}$$

which assures that the anti-self-dual part of $F'_{\mu\nu}$, being proportional to $\bar{\eta}^a_{\mu\nu} V_1^\dagger \sigma_a X_1 V_1$, vanishes. We emphasize that the projection $F \mapsto F^p$ is not used explicitly in this computation but nevertheless required if we want F' to come from a gauge potential A' which derives from A^p by an MvN transformation.

Obviously, the MvN transformation (4.21) extends the self-duality of $F^p_{\mu\nu}$ to the whole Fock space $\mathcal{H} \otimes \mathbb{C}^2$. This is exactly what was desired to cure the incompleteness of the configuration (3.20) with ϕ from (3.28). In combined form, our complete one-instanton gauge field reads

$$F'_{\mu\nu} = \frac{1}{r} \begin{pmatrix} z_\epsilon & \bar{y} S_1^\dagger \\ y & -\bar{z}_\epsilon S_1^\dagger \end{pmatrix} \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\Big) \begin{pmatrix} \bar{z}_\epsilon & \bar{y} \\ S_1 y & -S_1 z_\epsilon \end{pmatrix} \frac{1}{r} \tag{4.25}$$

with abbreviations from (3.28), (3.29), and (4.16). Here, $A_\mu = A_\mu(\phi_1)$ is obtained by substituting (3.28) into (3.20).

In the commutative limit, our configuration (A', F') clearly reduces to the BPST instanton because $\check{S}_1 = \operatorname{diag}(1, S_1)$ becomes a unitary matrix and the remaining part of the MvN transformation (mediated by U_1) turns *exactly* into the singular gauge transformation from the 't Hooft to the BPST gauge (see [48]). For illustration, the operator S_1 from (4.17) can be rewritten as (cf. [7])

$$S_1 = \bar{y}\,(\bar{y}y + 2\theta)^{-1} y + (1 - \bar{y}\,(\bar{y}y + 2\theta)^{-1} y)\, \bar{z}_\epsilon\, (\bar{z}_\epsilon z_\epsilon + 2\theta)^{-1/2}\,, \tag{4.26}$$

from which one easily sees that in the commutative limit it approaches the identity.

Multi-instanton solutions. Again, a projection (by $1{-}P_{n-1}$) plus a Murray-von Neumann transformation (by V_n) will enable us to extend the projected self-dual solution to the complete Fock space. We decompose $V_n = \check{S}_n\, U_n$ and propose

$$\check{S}_n = \begin{pmatrix} 1 & 0 \\ 0 & S_n \end{pmatrix} \qquad \text{and} \qquad U_n = \mathcal{U}\, U_1^n\, \mathcal{U}^\dagger\,, \tag{4.27}$$

where

$$\mathcal{U} := \sum_{i\geq 0} |h_{i+1}\rangle\langle\hat{i}| \tag{4.28}$$

is the unitary transformation from the basis $\{|\hat{i}\rangle\}$ to the basis $\{|h_{i+1}\rangle\}$ in $\mathcal{H}$, and $S_n : \mathcal{H} \to (1-P_{n-1})\mathcal{H}$ is to satisfy the relations

$$S_n^\dagger\, S_n = 1 \qquad \text{and} \qquad S_n\, S_n^\dagger = 1 - P_{n-1} = 1 - \sum_{i=1}^{n} |h_i\rangle\langle h_i|\,. \tag{4.29}$$

In the orthonormal basis one can choose S_n and $S_n^\dagger$ in the form

$$S_n = \sum_{i\geq 1} |h_{i+n}\rangle\langle h_i| \qquad \text{and} \qquad S_n^\dagger = \sum_{i\geq 1} |h_i\rangle\langle h_{i+n}|\,. \tag{4.30}$$

Since this basis was constructed from the states $|b_1\rangle, \ldots, |b_n\rangle$, the operator S_n contains all the information about the position parameters $b_1, \ldots, b_n$. It is easy to see that

$$S_n = \mathcal{U}\, S_1^n\, \mathcal{U}^\dagger \tag{4.31}$$

with S_1 from (4.20).

Let us justify our proposal (4.27). Employing the relations

$$U_1^n = 1\; - \sum_{i=0}^{n-1} \begin{pmatrix} |\hat{i}\rangle\langle\hat{i}| & 0 \\ 0 & 0 \end{pmatrix} \tag{4.32}$$

with

$$\mathcal{U} \left(\sum_{i=0}^{n-1} |\hat{i}\rangle\langle\hat{i}| \right) \mathcal{U}^\dagger = \sum_{i=0}^{n-1} |h_{i+1}\rangle\langle h_{i+1}| = P_{n-1} \tag{4.33}$$

one can show that

$$U_n^\dagger\, U_n = 1 \otimes \mathbf{1}_2 \qquad \text{but} \qquad U_n\, U_n^\dagger = \begin{pmatrix} 1-P_{n-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.34}$$

Together with (4.29) one gets

$$V_n^\dagger\, V_n = 1 \otimes \mathbf{1}_2 \qquad \text{but} \qquad V_n\, V_n^\dagger = \begin{pmatrix} 1-P_{n-1} & 0 \\ 0 & 1-P_{n-1} \end{pmatrix} = (1 - P_{n-1}) \otimes \mathbf{1}_2\,. \tag{4.35}$$

Finally, substituting (3.26) into (3.20), computing its field strength $F_{\mu\nu}$ and performing the (extended) MvN transformation (4.11) by the matrices (4.27) we learn that the anti-self-dual part of $F'_{\mu\nu}$ indeed vanishes everywhere in $\mathcal{H} \otimes \mathbb{C}^2$, again due to $X_n V_n = 0 = V_n^\dagger X_n$. Hence, our final result

$$A'_\mu = V_n^\dagger\, A_\mu^p\, V_n + V_n^\dagger\, \partial_\mu\, V_n \qquad \text{and} \qquad F'_{\mu\nu} = V_n^\dagger\, F_{\mu\nu}^p\, V_n \;\equiv\; V_n^\dagger\, F_{\mu\nu}\, V_n \tag{4.36}$$

constitutes a proper non-commutative generalization of the 't Hooft n-instanton solution. It has topological charge $Q = n$ since in the $\theta \to 0$ limit our solution coincides with the standard 't Hooft solution, and the topological charge does not depend on θ. This may also be shown by reducing the action integral to the trace of the projector P_{n-1}, following Furuuchi [11].

5. Concluding remarks

Proper non-commutative instantons are constructed by not only replacing the coordinates in the commutative configuration with their operator analogues but also applying a projection and an appropriate MvN transformation. We have demonstrated this beyond the (previously considered) case of the one-instanton solution, providing explicit formulae for the field strength (but not the gauge potential) of regular non-commutative 't Hooft multi-instantons in U(2). Our results are easily generalized to U(N). We have pointed out that the MvN transformation is needed to remove the source singularities in the reduced SDYM equation, which hamper self-duality on some subspace. In order to take advantage of the MvN transformation, however, we had to pass from the original gauge potential to a "projected" gauge potential in an implicit manner, thereby foregoing an explicit solution for it.

We found it easy to work with the twistor approach because it and the Atiyah-Ward ansätze directly generalize to the non-commutative realm, providing us with a systematic and straightforward strategy for the construction of self-dual gauge-field configurations. In this method, the main task is to find two holomorphic (in λ and $\tilde{\lambda}$, respectively) regular matrix-valued operators ψ_+ and ψ_- such that their "ratio" $\psi_+^{-1}\psi_-$ defines a holomorphic bundle over non-commutative twistor space with appropriate global properties. In fact, finding solutions to the splitting problem is not made any harder by noncommutativity.

Multi-instantons can also be obtained by employing the dressing approach [50, 51]. In its non-commutative variant, one is to find a meromorphic (in λ) matrix-valued operator ψ (having finite-order poles in the spectral parameter λ) which obeys some linear differential equations. The non-commutative dressing method was successfully applied to the study of non-commutative solitons in a $2+1$-dimensional integrable field theory [52, 53]. It would be illuminating to also exercise it on the subject of non-commutative instantons and to compare the results with those obtained in the twistor approach.

Finally, non-commutative instantons are interpreted as Dp-branes within coincident D(p+4)-branes carrying a constant two-form B-field background [3]. Thus, they have immediate bearing on the issue of non-perturbative string backgrounds.

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A. Non-commutative 't Hooft instantons from the ADHM approach

ADHM construction. Here we restrict ourselves to the case of self-dual non-commutative euclidean space $\mathbb{R}^4_\theta$ with the tensor (3.2) given by

$$\theta^{\mu\nu} = \theta\,\eta^{3\mu\nu}\,, \tag{A.1}$$

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where

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon^a_{bc}\,, & \text{for } \mu = b\,,\ \nu = c\,, \\ \delta^a_\mu\,, & \text{for } \nu = 4\,, \\ -\delta^a_\nu\,, & \text{for } \mu = 4 \end{cases} \tag{A.2}$$

is the self-dual 't Hooft tensor.

Let us introduce the matrices

$$(e_\mu) = (-\mathrm{i}\sigma_a\,,\,1) \qquad \text{and} \qquad (e^\dagger_\mu) = (\mathrm{i}\sigma_a\,,\,1) \tag{A.3}$$

which enjoy the properties

$$\begin{aligned} e^\dagger_\mu\, e_\nu &= \delta_{\mu\nu} + \eta^a_{\mu\nu}\,\mathrm{i}\sigma_a \;=:\; \delta_{\mu\nu} + \eta_{\mu\nu}\,, \\ e_\mu\, e^\dagger_\nu &= \delta_{\mu\nu} + \bar\eta^a_{\mu\nu}\,\mathrm{i}\sigma_a \;=:\; \delta_{\mu\nu} + \bar\eta_{\mu\nu}\,. \end{aligned} \tag{A.4}$$

Using these matrices one can introduce $x := x^\mu e^\dagger_\mu$ with $\{x^\mu\} \in \mathbb{R}^4_\theta$.

The (modified) ADHM construction (see [4, 7, 9, 15, 20]) of an n-instanton solution is based on a $(2n{+}2)\times 2$ matrix Ψ and a $(2n{+}2)\times 2n$ matrix $\Delta = a + b(x\otimes\mathbf{1}_n)$ where a and b are constant $(2n{+}2)\times 2n$ matrices. These matrices must satisfy the following conditions:

$$\Delta^\dagger\Delta \qquad \text{is invertible}\,, \tag{A.5}$$

$$[\,\Delta^\dagger\Delta\,,\, e_\mu \otimes \mathbf{1}_n\,] \;=\; 0 \qquad \forall x\,, \tag{A.6}$$

$$\Delta^\dagger\Psi \;=\; 0\,, \tag{A.7}$$

$$\Psi^\dagger\Psi \;=\; \mathbf{1}_2\,. \tag{A.8}$$

It is not difficult to see that conditions (A.5) and (A.6) are met if

$$\Delta^\dagger\Delta = \mathbf{1}_2 \otimes f^{-1}_{n\times n}\,. \tag{A.9}$$

For (Δ, Ψ) satisfying (A.5)–(A.8) the gauge potential is chosen in the form

$$A = \Psi^\dagger\, d\Psi\,. \tag{A.10}$$

The resulting gauge field F will be self-dual if Δ and Ψ obey the completeness relation

$$\Psi\,\Psi^\dagger + \Delta\,(\Delta^\dagger\Delta)^{-1}\Delta^\dagger = \mathbf{1}_{2n+2}\,. \tag{A.11}$$

Ansatz. For constructing non-commutative 't Hooft n-instantons let us take (cf. [54, 55])

$$\Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}, \qquad a = \begin{pmatrix} \Lambda_1\mathbf{1}_2 & \dots & \Lambda_n\mathbf{1}_2 \\ -b_1 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & -b_n \end{pmatrix}, \qquad \text{and} \qquad b = \begin{pmatrix} \mathbf{0}_2 & \dots & \mathbf{0}_2 \\ \mathbf{1}_2 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & \mathbf{1}_2 \end{pmatrix} \tag{A.12}$$

with $b_i = b^\mu_i e^\dagger_\mu$. It follows that

$$\Delta^\dagger\Delta = \mathbf{1}_2 \otimes (\delta_{ij} r_j^2 + \Lambda_i\Lambda_j) \;=:\; \mathbf{1}_2 \otimes (R + \Lambda\Lambda^T)\,, \tag{A.13}$$

where

$$R = \begin{pmatrix} r_1^2 & & 0 \\ & \ddots & \\ 0 & & r_n^2 \end{pmatrix}, \qquad \Lambda = (\Lambda_1, \dots, \Lambda_n), \qquad r_j^2 = \delta_{\mu\nu}\,(x^\mu - b_j^\mu)(x^\nu - b_j^\nu), \tag{A.14}$$

and the Λ_i are constants parametrizing the scale of the ith instanton. From (A.13) we see that the condition (A.9) (and thus also (A.5) and (A.6)) is satisfied. Indeed, by direct calculation one finds

$$\mathbf{1}_2 \otimes f_{n\times n} = (\Delta^\dagger \Delta)^{-1} = \mathbf{1}_2 \otimes (R^{-1} - R^{-1}\Lambda\, \phi_n^{-1} \Lambda^T R^{-1}) \tag{A.15}$$

with

$$\phi_n = 1 + \sum_{i=1}^{n} \frac{\Lambda_i^2}{r_i^2}\,. \tag{A.16}$$

For the given $\Delta = a + b(x \otimes \mathbf{1}_n)$ and $x_i := (x^\mu - b_i^\mu) e_\mu^\dagger$, the condition (A.7) becomes

$$\Lambda_i \Psi_0 + x_i^\dagger \Psi_i = \mathbf{0}_2\,, \qquad \text{for } i = 1, \dots, n\,. \tag{A.17}$$

These equations are solved by

$$\Psi_0 = \phi_n^{-1/2}\, \mathbf{1}_2 \qquad \text{and} \qquad \Psi_i = -x_i \frac{\Lambda_i}{r_i^2}\, \phi_n^{-1/2} \tag{A.18}$$

where the factor $\phi_n^{-1/2}$ was introduced to achieve the normalization

$$\Psi^\dagger \Psi = \phi_n^{-1/2} \left(1 + \sum_{i=1}^{n} \frac{\Lambda_i^2}{r_i^2} \right) \mathbf{1}_2\, \phi_n^{-1/2} = \mathbf{1}_2\,. \tag{A.19}$$

Hence, our (Δ, Ψ) satisfies all conditions (A.5)–(A.9), and we can present the gauge potential (A.10).

Completeness relation. Being convinced that they have constructed self-dual field configurations, many authors stop at this point and ignore the completeness relation (A.11). However, the latter may be violated, in which case one obtains a *singular* field configuration (cf. the discussion in [20]). Indeed, our example solution (A.18) indicates just that. Namely, after (lengthy) computations we arrive at

$$\Psi\, \Psi^\dagger + \Delta\, (\Delta^\dagger \Delta)^{-1} \Delta^\dagger = \mathbf{1}_{2n+2} - \Pi\,, \tag{A.20}$$

where Π is a projector,

$$\Pi := \begin{pmatrix} \mathbf{0}_2 & & & \mathbf{0}_2 \\ & \mathbf{1}_2 - x_1 \frac{1}{r_1^2} x_1^\dagger & & \\ & & \ddots & \\ \mathbf{0}_2 & & & \mathbf{1}_2 - x_n \frac{1}{r_n^2} x_n^\dagger \end{pmatrix},$$

$$\text{and} \qquad \mathbf{1}_2 - x_i \frac{1}{r_i^2} x_i^\dagger = \begin{pmatrix} |b_i\rangle\langle b_i| & 0 \\ 0 & 0 \end{pmatrix}. \tag{A.21}$$

Here, the $|b_i\rangle$ are the coherent states defined in (3.32). Notice that in the commutative limit $\Pi \to 0$, and the completeness relation will be saturated.

Field strength. We finally evaluate the field strength. Substituting (A.18) into (A.10) and using (A.20) we find

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu(\Psi^\dagger\partial_\nu\Psi) - \partial_\nu(\Psi^\dagger\partial_\mu\Psi) + [\,\Psi^\dagger\partial_\mu\Psi\,,\,\Psi^\dagger\partial_\nu\Psi\,] \\
&= (\partial_\mu\Psi^\dagger)(1-\Psi\Psi^\dagger)\,\partial_\nu\Psi - (\mu\leftrightarrow\nu) \\
&= (\partial_\mu\Psi)^\dagger(\Delta\,(\Delta^\dagger\Delta)^{-1}\Delta^\dagger + \Pi)\,\partial_\nu\Psi - (\mu\leftrightarrow\nu) \\
&= \Psi^\dagger(\partial_\mu\Delta)\,(\Delta^\dagger\Delta)^{-1}(\partial_\nu\Delta^\dagger)\Psi + (\partial_\mu\Psi)^\dagger\,\Pi\,\partial_\nu\Psi \;-\; (\mu\leftrightarrow\nu) \\
&= \Psi^\dagger\,b\,e_\mu^\dagger(\Delta^\dagger\Delta)^{-1}e_\nu\,b^\dagger\Psi + \phi_n^{-1/2}e_\mu\left(0,-\frac{\Lambda_1}{r_1^2},\ldots,-\frac{\Lambda_n}{r_n^2}\right)\Pi\begin{pmatrix}0\\ -\frac{\Lambda_1}{r_1^2}\\ \vdots\\ -\frac{\Lambda_n}{r_n^2}\end{pmatrix}e_\nu^\dagger\,\phi_n^{-1/2} - \\
&\quad -(\mu\leftrightarrow\nu) \\
&= 2\,\Psi^\dagger b\,(\Delta^\dagger\Delta)^{-1}\eta_{\mu\nu}\,b^\dagger\Psi + \phi_n^{-1/2}\left(\sum_{i=1}^n\frac{\Lambda_i^2}{4\theta^2}\,|b_i\rangle\langle b_i|\right)\phi_n^{-1/2}\,2\,\bar\eta_{\mu\nu} \\
&= 2\,\Psi^\dagger b\,(\Delta^\dagger\Delta)^{-1}\eta_{\mu\nu}\,b^\dagger\Psi - X_n\,2\,\bar\eta_{\mu\nu}\,,
\end{aligned}
\tag{A.22}
$$

where X_n is precisely the source term (3.35) derived earlier in the twistor approach. We conclude that the anti-self-dual part of $F_{\mu\nu}$ is nonzero and, in complex coordinates, coincides with the r.h.s. of (3.23). Hence, the ADHM approach encounters exactly the same obstacle as the twistor method. As shown in section 4 the remedy of this deficiency consists in a particular projection of the gauge potential followed by a suitable Murray-von Neumann transformation.

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